

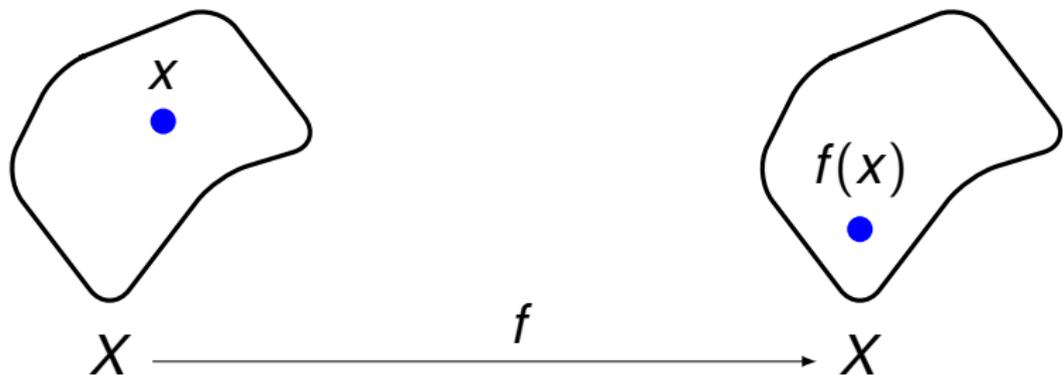
Shadowing in CR-Dynamical Systems

Andrew Wood

Topological Dynamical Systems

Definition

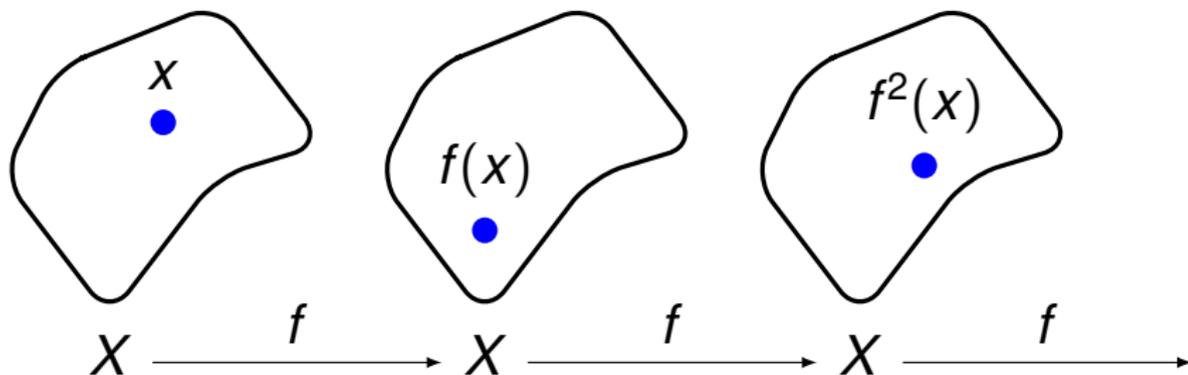
For a non-empty compact metric space X and continuous self-map $f : X \rightarrow X$, we say (X, f) is a *topological dynamical system*.



Topological Dynamical Systems

Definition

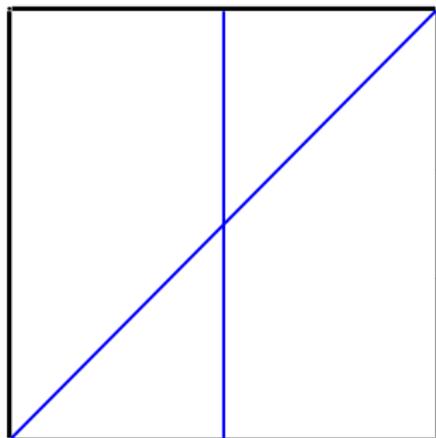
Let (X, f) be a topological dynamical system. Then the *trajectory* of a point $x \in X$ is the sequence $\langle x, f(x), f^2(x), \dots \rangle$.



CR-Dynamical Systems

Definition

For a non-empty compact metric space X and non-empty closed $G \subseteq X \times X$, we say (X, G) is a *CR-Dynamical System*.



$$X = [0, 1]$$

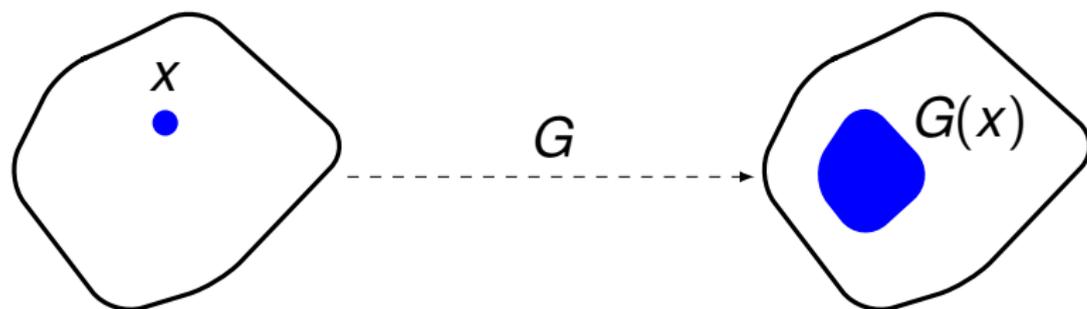
$$G = \Delta_X \cup \left(\left\{ \frac{1}{2} \right\} \times X \right)$$

Notation

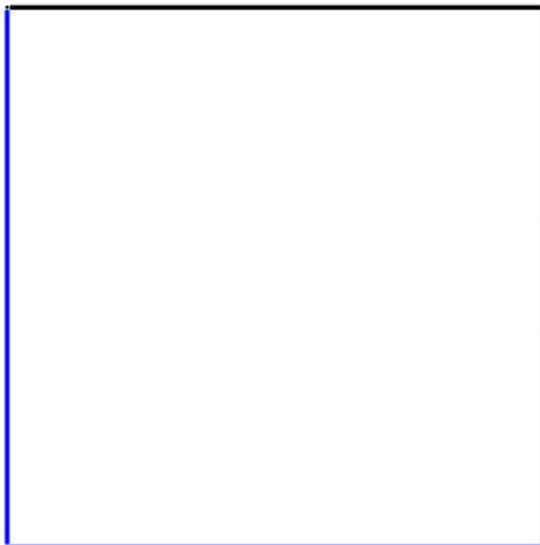
Definition

Let (X, G) be a CR-dynamical system and $x \in X$. Then we define

- $G(x) = \{y \in X \mid (x, y) \in G\}$;
- $G^n(x) = \bigcup_{y \in G^{n-1}(x)} G(y)$ for each $n > 1$; and
- $G^n = \{(x, y) \in X \times X \mid y \in G^n(x)\}$.



Example



$$X = [0, 1]$$

$$G = X \times \{0\} \cup \{0\} \times X$$

$$G\left(\frac{1}{2}\right) = \{0\}$$

$$G^2\left(\frac{1}{2}\right) = X$$

Trajectories

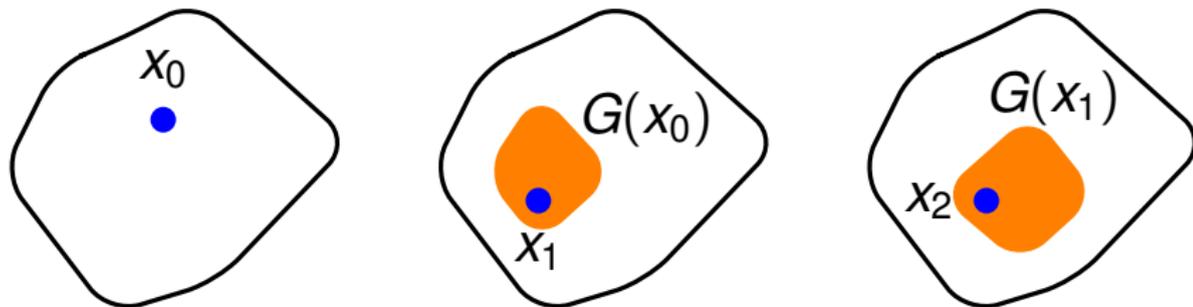
Definition

Let (X, G) be a CR-dynamical system. Then a *trajectory* of a point $x \in X$ is a sequence

$\langle x_n \mid n \in \mathbb{N} \rangle$ such that

- $x_0 = x$; and
- $x_{n+1} \in G(x_n)$ for all $n \in \mathbb{N}$.

Denote by $T_G^+(x)$ the set of all trajectories of x .

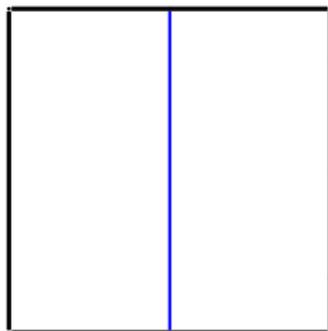


Legality

Definition

Suppose (X, G) is a CR-dynamical system.

- We say $x \in X$ is *legal* if $T_G^+(x) \neq \emptyset$
- We say $x \in X$ is *illegal* if $T_G^+(x) = \emptyset$.
- Denote by $\text{legal}(G)$ the set of legal points.



$$X = [0, 1]$$

$$G = \left\{ \frac{1}{2} \right\} \times X$$

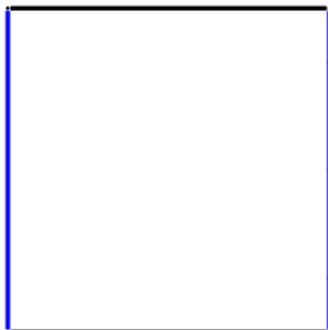
$$\text{legal}(G) = \left\{ \frac{1}{2} \right\}$$

Domain

Definition

Suppose (X, G) is a CR-dynamical system. The *domain* of G is the set

$$\text{dom}(G) = \{x \in X \mid G(x) \neq \emptyset\}.$$



$$X = [0, 1]$$

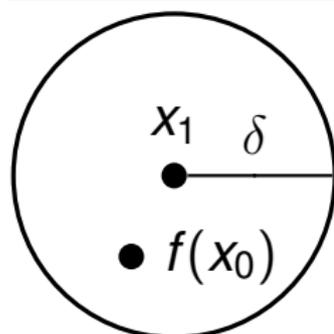
$$G = \{0, 1\} \times X$$

$$\text{dom}(G) = \{0, 1\}$$

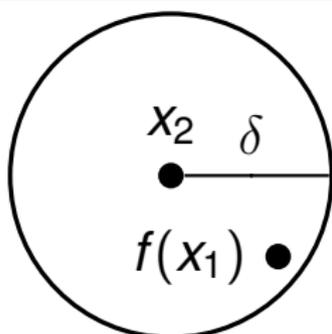
δ -Pseudo-Orbits

Definition

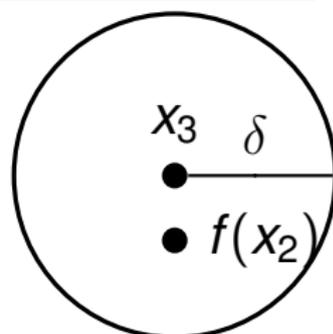
Let (X, f) be a topological dynamical system and $\delta > 0$. A sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ is a *δ -pseudo-orbit*, provided $d(f(x_n), x_{n+1}) \leq \delta$ for each $n \in \mathbb{N}$.



$B[x_1, \delta]$



$B[x_2, \delta]$



$B[x_3, \delta]$

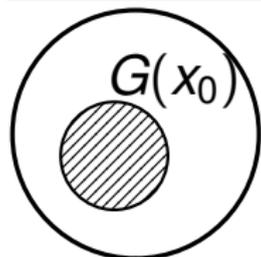
$(\delta, 1)$ -Pseudo-Orbits

Definition

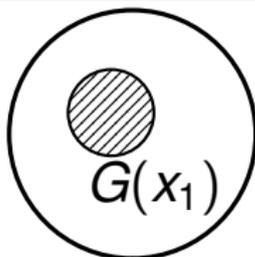
Let (X, G) be a CR-Dynamical System and $\delta > 0$. We say a sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ in $\text{dom}(G)$ is a (δ, i) -pseudo-orbit, provided $d(x_{n+1}, y) \leq \delta$ for

- $(i = 1)$ every $y \in G(x_n)$,

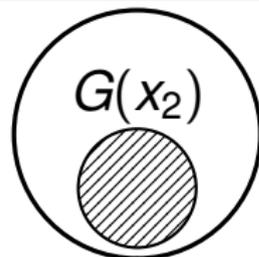
for all $n \in \mathbb{N}$.



$B[x_1, \delta]$



$B[x_2, \delta]$



$B[x_3, \delta]$

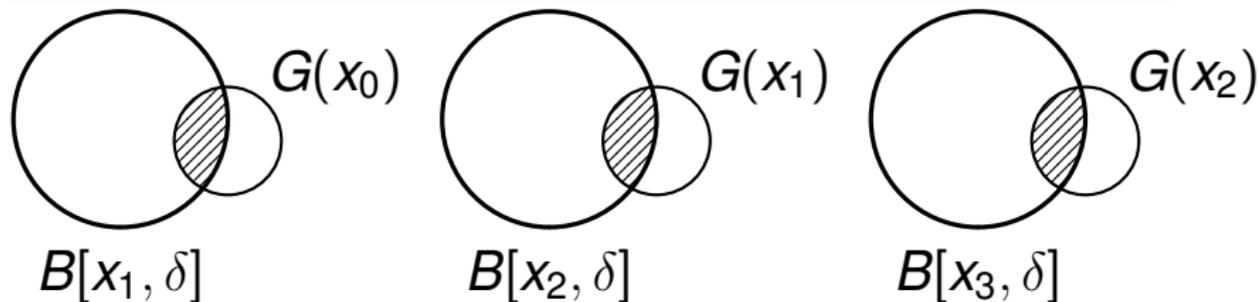
$(\delta, 2)$ -Pseudo-Orbits

Definition

Let (X, G) be a CR-Dynamical System and $\delta > 0$. We say a sequence $\langle x_n \mid n \in \mathbb{N} \rangle$ in $\text{dom}(G)$ is a (δ, i) -pseudo-orbit, provided $d(x_{n+1}, y) \leq \delta$ for

- $(i = 1)$ every $y \in G(x_n)$,
- $(i = 2)$ some $y \in G(x_n)$,

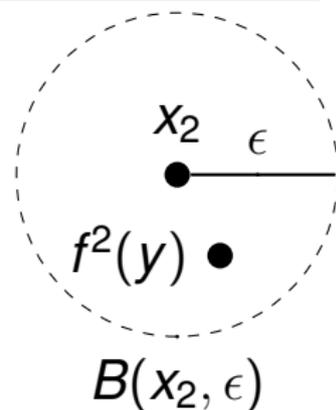
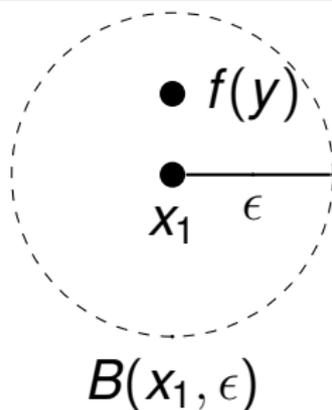
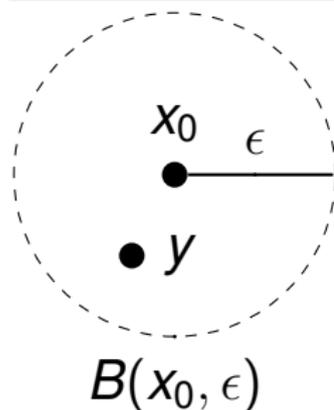
for all $n \in \mathbb{N}$.



ϵ -Shadowing-Points

Definition

Let (X, f) be a topological dynamical system and $\epsilon > 0$. A point $y \in X$ **ϵ -shadows** a sequence $\langle x_n \mid n \in \mathbb{N} \rangle$, provided $d(f^n(y), x_n) < \epsilon$ for each $n \in \mathbb{N}$.



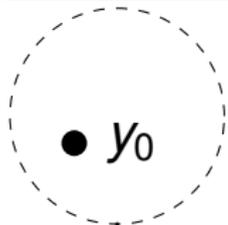
(ϵ, j) -Shadowing-Points

Definition

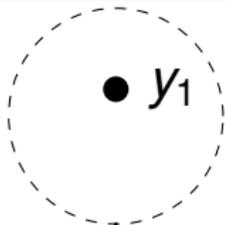
Let (X, G) be a CR-dynamical system and $\epsilon > 0$. We say $y \in \text{legal}(G)$ (ϵ, j) -*shadows* a sequence $\langle x_n \mid n \in \mathbb{N} \rangle$, provided for

- $(j = 1)$ **all** $\langle y_n \mid n \in \mathbb{N} \rangle \in T_G^+(y)$,
- $(j = 2)$ **some** $\langle y_n \mid n \in \mathbb{N} \rangle \in T_G^+(y)$,

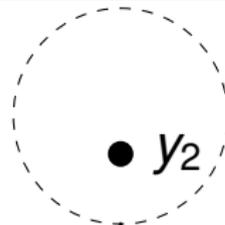
we have $d(x_n, y_n) < \epsilon$ for each $n \in \mathbb{N}$.



$B(x_0, \epsilon)$



$B(x_1, \epsilon)$



$B(x_2, \epsilon)$

The Shadowing Properties

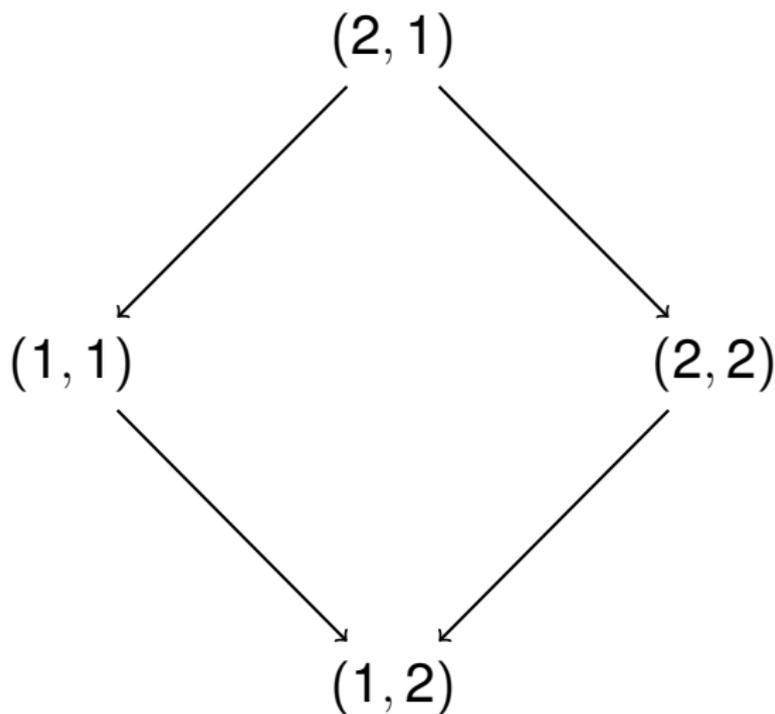
Definition

A topological dynamical system (X, f) has the *shadowing property*, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that every δ -pseudo-orbit is ϵ -shadowed by a point in X .

Definition

A CR-dynamical system (X, G) has the *(i, j) -shadowing property*, if for every $\epsilon > 0$ there exists a $\delta > 0$ such that every (δ, i) -pseudo-orbit is (ϵ, j) -shadowed by a point in X .

How are they related?



Generalisation

Theorem

Let (X, f) be a topological dynamical system. Then (X, f) has the shadowing property if, and only if, (X, f^n) has the shadowing property for all $n \geq 1$.

Theorem

Let (X, G) be a CR-dynamical system, and $j \in \{1, 2\}$. Then (X, G) has the $(2, j)$ -shadowing property if, and only if, (X, G^n) has the $(2, j)$ -shadowing property for all $n \geq 1$.

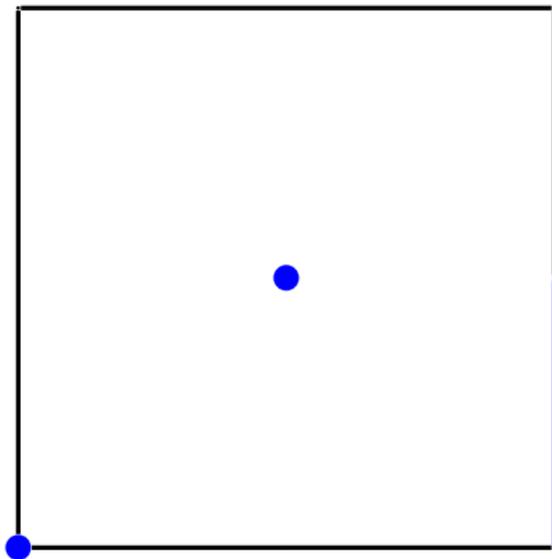
Characterisation when G is finite

Theorem

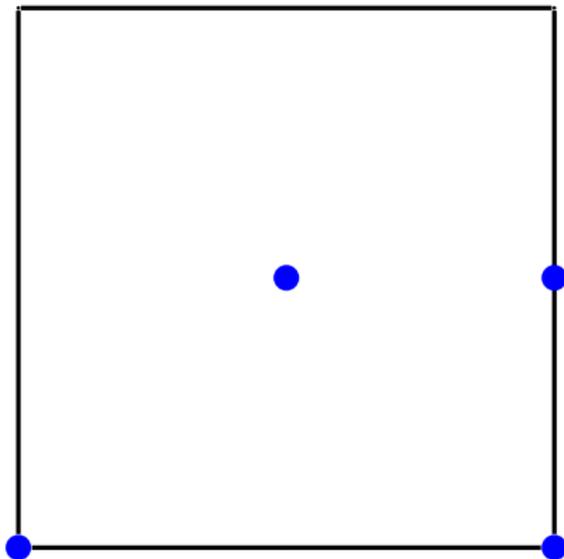
Let (X, G) be a CR-dynamical system, with G finite. Then,

- *(X, G) has the $(1, 1)$, $(1, 2)$ and $(2, 2)$ shadowing properties, but need not have the $(2, 1)$ -shadowing property;*
- *(X, G) has the $(2, 1)$ -shadowing property if, and only if, each $x \in \text{legal}(G)$ has exactly one trajectory.*

Example



$$G = \left\{ (0, 0), \left(\frac{1}{2}, \frac{1}{2} \right) \right\} \cup \left(\{1\} \times [0, \frac{1}{2}] \right)$$



$$G^2 = \left\{ (0, 0), \left(\frac{1}{2}, \frac{1}{2} \right), (1, 0), \left(1, \frac{1}{2} \right) \right\}$$

Where to from here?

- Analyse shadowing in interesting examples, such as extensions of tent maps, the Cantor fan, and the Lelek fan.
- See how the shadowing properties interact with other dynamical properties, such as transitive points and chain transitivity.
- Generalise results about shadowing from topological dynamical systems.